PETRI MAP FOR RANK TWO BUNDLES WITH CANONICAL DETERMINANT

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ABSTRACT. We prove the Bertram-Feinberg-Mukai conjecture for a generic curve C of genus g and a semistable vector bundle E of rank two and determinant K on C, namely we prove the injectivity of the Petri-canonical map $S^2(H^0(E)) \to H^0(S^2(E))$.

1. INTRODUCTION

Consider a generic curve of genus g. Classical Brill-Noether Theory deals with the existence of line bundles of a fixed degree d that possess a preassigned number of sections k traditionally denoted by $r+1$. The loci of such line bundles L is a subvariety of the Picard scheme $Pic^d(C)$ and is usually denoted by W_d^r.

A key role in the study of classical Brill-Noether Theory is played by the Petri map: Fix a line bundle L such that $h^0(C, L) = r + 1$. Consider the map

$$H^0(C, L) \otimes H^0(C, \omega \otimes L^{-1}) \to H^0(C, \omega).$$

The orthogonal to the image of this map is identified with the tangent space to W_d^r at the point L. For generic C and every L, this map is injective. This fact alone proves that W_d^r is empty when its expected dimension is negative and of expected dimension when nonempty. Moreover, its singular locus is W_d^{r+1}.

The most straightforward generalization of Brill-Noether Theory to higher rank is to consider the loci inside the moduli space of stable vector bundles of rank r and degree d that have at least a given number k of sections. The equivalent of the Petri map is then

$$H^0(C, E) \otimes H^0(C, \omega \otimes E^*) \to H^0(C, \omega \otimes E \otimes E^*).$$

The vector space on the right hand side is identified to the cotangent space to the moduli space of vector bundles of fixed rank and degree (but variable determinant). As in the line bundle case, the orthogonal to the image of this map is the tangent space at E to the locus of vector bundles with k sections. These loci, though, don't enjoy most of the good properties of classical Brill-Noether theory, in particular the Petri

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map is not injective for many stable vector bundles even on a generic curve (see [T1]).

On the other hand if one restricts to the case of rank two with fixed determinant the canonical line bundle ω , one obtains the moduli space $U(2,\omega)$. Bertram, Feinberg and Mukai (cf.[BF], [M1]) introduced the loci $B_{2,\omega}^k$ of (semi)stable vector bundles E of rank 2 and determinant ω that have at least k independent sections. They conjectured, based on some evidence for small genus, that these loci would be well behaved. The object of this paper is to prove the part of this conjecture about non-existence and singularity. The question of existence was partially answered in our previous work [T3] as well as in [P].

We want to study the equivalent of the Petri map in this situation. When the rank is two and the determinant $\wedge^2 E \cong \omega$, there is a natural identification of $\omega \otimes E^* \cong E$. Also, the tangent space to $U(2,\omega)$ inside the moduli space of stable vector bundles of rank two and degree $2g-2$ can be identified to the dual of $H^0(C, S^2(E))$ where $S^2(E)$ denotes the symmetric power of E and is interpreted as a quotient of $E \otimes E$. As $\wedge^2 H^0(E)$ maps to zero when passing to the quotient, we can factor the map

$$H^0(C,E) \otimes H^0(C,E) \rightarrow H^0(C, S^2(E))$$

by moding out the original space $H^0(C,E) \otimes H^0(C,E)$ by $\wedge^2 H^0(E)$. One obtains a map that we shall call the canonical Petri map or simply the Petri map when there is no danger of confusion:

$$S^2 H^0(C,E) \rightarrow H^0(S^2 E).$$

Then the tangent space to $B_{2,\omega}^k$ at E is identified with the orthogonal to the image of this map. Hence, its injectivity shows that the locus is empty when the expected dimension $\rho = \rho_{2,\omega}^k = 3g - 3 - \binom{k+1}{2} < 0$, of dimension exactly ρ when non-empty and singular only along $B_{2,\omega}^{k+1}$. In particular, when $\rho = -1$, the generic curve does not have a linear series with so many sections and the locus of curves where the sections exist is expected to be a divisor of the moduli space of curves $\mathcal{M}_g$. When $g = 10$ this divisor provided the first counterexample to the slope conjecture (see [FP]). For various values of k, g for which $\rho = -1$, one can now obtain divisors in $\mathcal{M}_g$ that can provide information on the slope of $\mathcal{M}_g$.

In addition, from work of Mukai [M2], these loci have been useful in studying which curves are contained in $K3$ surfaces. For curves of small genus and for particular values of k, their geometry seems very interesting [OPP] .

Our result is:

1.1. Theorem *The canonical Petri map is injective for every semistable vector bundle E of rank two and determinant ω on a generic curve of genus g defined over an algebraically closed field of characteristic different from two.*

The proof is based on some of the techniques used to show the injectivity of the classical Petri map in [EH], although dealing with vector bundles of rank two is substantially more involved than dealing with line bundles. The degeneration of the smooth curve to a singular reducible curve is also different. Instead of a rational curve with g elliptic tails as in [EH], we consider a chain of elliptic curves (see [W]). This allows us to work in arbitrary characteristic different from two. Moreover, as pointed out in [FP] the result fails for at least some of the Eisenbud-Harris curves.

We will first need to determine what are the possible limits of vector bundles with canonical determinant when the curve becomes reducible. The limit of the vector bundle may become a torsion-free sheaf for which the determinant is not even defined. We use the results of Nagaraj-Seshadri ([NS]) as well as the results in [T2] to find these limits. Then the theory of limit linear series for vector bundles ([T1]) can be used to study the behavior of sections in the kernel.

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2. Conditions on semistable vector bundle with canonical determinant on reducible curves

Let C be a reducible curve of arithmetic genus g consisting of two components C_1, C_2 with one node obtained by gluing the point $Q_1 \in C_1$ to the point $P_2 \in C_2$. Fix a polarization of C (i.e. positive integers w_1, w_2 such that $w_1 + w_2 = 1$). Fix a degree d for rank two bundles and define $\chi = d + 2(1 - g)$. If $w_1\chi$ is not an integer, then the moduli space of rank two bundles on C that are semistable by the given polarization consists of two components that are characterized by the splitting of the degrees among C_1 and C_2 (see [T2]). Define the two numbers $\chi_1 = \chi(E_{|C_1})$ and $\chi_2 = \chi(E_{|C_2})$ where χ denotes the Euler-Poincaré characteristic. Then, the semistability condition gives

$$(*) w_1\chi \leq \chi_1 \leq w_1\chi + 2.$$

We now look at the conditions on the determinant Let d_1 be such that $\chi_1 = d_1 + 2(1 - g)$ is the lowest possible value allowed in (*). Let d_2 be defined as $d_2 = d - d_1$. One can then define the determinant

map from the moduli space of torsion free sheaves on C to the Picard variety of line bundles of bidegree d_1, d_2 on C as follows ([NS]):

a) If E is a vector bundle of bidegree d_1, d_2, take its determinant.

b) If E is a vector bundle of bidegree $d_1 + 1, d_2 - 1$, take

$$((detE_1)(-Q_1), (detE_2)(P_2)).$$

c) If E is a sheaf that is not locally free, then at the node P (see 2.3 b) below), $E_P \equiv \mathcal{O}_P \oplus \mathcal{M}_P$. In this case, $w_1\chi < \chi_1 < w_1\chi + 1$ and $\chi_1 + \chi_2 = \chi + 1$. Hence, the bidegree is $d_1, d_2 - 1$. The image of E by the determinant map is then

$$(det(E_1), det(E_2(P_2)).$$

These results will be generalized in 2.2 to reducible curves with more components.

In the rest of the paper we shall consider a curve of the following type:

2.1. Definition *Assume we have curves $C_1, ..C_M$ each with points P_i, Q_i. Glue Q_i to $P_{i+1}, i = 1..M - 1$. Assume that every curve is either rational or elliptic, the points P_i, Q_i are generic on the (elliptic) curve and the genus of the resulting curve is g (and there are therefore exactly g elliptic components). We shall call such a curve a chain of genus g*

Assume that a chain of genus g as above is the central fiber of a one-dimensional family of curves. If we modify the total space by base change and blow-ups, the central fiber will still be of the same kind, may be with a few more rational components.

2.2. Claim. *Fix a multidegree $d_1, ..d_M$ giving rise to a component of the moduli space of torsion free sheaves on the total curve C semistable by a certain polarization. Fix another component of the moduli space of vector bundles of rank two on C corresponding to a multidegree $d_1 + \epsilon_1, ..d_M + \epsilon_M$ (hence $\sum \epsilon_i = 0$). Let E be an element of this component with restriction to C_i being E_i . Then the determinant map takes the form*

$$E \to (detE_i)((\sum_{k<i} \epsilon_k)P_i - (\sum_{k\leq i} \epsilon_k)Q_i)$$

Proof. We use induction on M. If $M = 2$, this is the result of Nagaraj-Seshadri about the determinant map. Assume that the formula is correct for $M - 1$ and prove it for M. Consider our curve as the union of two pieces, C_M and the rest. Using the case of two components, we obtain that on the curve C_M the determinant must be taken to be

$(det E_M)(-\epsilon_M P_M)$. As $\sum_{i=1...M} \epsilon_i = 0$, one finds $\sum_{k<M} \epsilon_k = -\epsilon_M$ and $\sum_{k \leq M} \epsilon_k = 0$, this agrees with the formula above.

On the union of $C_1, ..C_{M-1}$, one must take the determinant that we had in the case of $M-1$ and modify it with $\epsilon_M Q_{M-1} = (-\sum_{k<M} \epsilon_k)Q_{M-1}$. This modification does not change anything except on C_{M-1}. On C_{M-1}, the restriction of the determinant for the case of $M-1$ components does not have Q_{M-1} in the expression as this is not a node of C_{M-1}. The expression then is $det(E_{M-1})((\sum_{k<M-1} P_k))$. Modifying this by $\epsilon_M Q_{M-1} = -(\sum_{k<M} \epsilon_k)Q_{M-1}$ gives the stated result.

$\square$

Assume given a family of curves C_t with generic curve non-singular and special curve $C_0 = C$ reducible. Consider a family of line bundles L_t degenerating to a line bundle L_0. Then the moduli space of vector bundles on C_t with determinant L_t degenerates to the set of torsion free sheaves on C_0 with determinant L_0 in the sense above (see [NS]).

In addition to the conditions on the degree of the restriction of the vector bundle to each component as stated in (*), one finds conditions on the structure of the restrictions of the vector bundles themselves. The statement that follows tells us that these restrictions cannot be too unstable. Its proof is very similar to the proof of (*) given in [T2] (see also [G])

2.3. Claim *Let C be a curve as in 2.1. Consider torsion-free sheaves of rank two and fixed Euler-Poincaré characteristic χ and a polarization $\{w_i\}$ such that $\sum_{i \in I} w_i \chi$ is not an integer for any $I \subset \{1, \ldots M\}$.*

a) The restriction of a semistable sheaf to one of the irreducible components is either indecomposable or the direct sum of two line bundles whose degrees differ in at most one unit (or of necessarily the same degree in the case of the first and last components).

b) If a sheaf is not locally free at one node P, then the fiber at that node is $E_P \cong \mathcal{M}_P \oplus \mathcal{O}_P$ where $\mathcal{M}_P$ denotes the maximal ideal at P.

Proof. Note that on a rational curve, every vector bundle is a direct sum of line bundles while on an elliptic curve a vector bundle of rank two is either indecomposable or the direct sum of two line bundles (see [A]). The only statement that needs to be proved in a) is that the difference between the degrees of these bundles is either one or zero and that the former is only a possibility for components with two nodes.

Assume that the restriction of E to a component C_i is the direct sum of two line bundles L', L''.

Consider the subsheaf F of E consisting of sections of L' on C_i vanishing at the nodes extended by zero outside C_i. Then, the semistability

condition is
$$\frac{\chi(L') - s}{w_i} = \frac{\chi(F)}{w_i} \le \frac{\chi(E)}{2}$$
where s denotes the number of nodes in the component.

Similarly, consider the subsheaf G consisting of sections of L'' on C_i and sections of $E_{|C_j}$ that glue with L'' on the components other than C_i. Then,
$$\frac{\chi(G)}{2 - w_i} = \frac{\chi(E) - \chi(L')}{2 - w_i} \le \frac{\chi(E)}{2}.$$
The two inequalities put together give
$$w_i \frac{\chi(E)}{2} \le \chi(L') \le w_i \frac{\chi(E)}{2} + s.$$
The same inequality holds for L''. By assumption, $w_i \chi$ is not an integer, hence, a) follows.

Assume now that the sheaf is not locally free at a node P and that the torsion-free sheaf is isomorphic at the node to $\mathcal{M}_P^2$. Denote by C_1', C_2' the two connected components of $C - \{P\}$. Consider the subsheaf F of E consisting of sections of E on C_1' vanishing at P extended by zero on C_2'. Write w_1' (w_2') for the sum of the w_i corresponding to C_1' (C_2'). Then, the semistability condition is
$$\frac{\chi(E_{|C_1'})}{2w_1'} = \frac{\chi(F)}{2w_1'} \le \frac{\chi(E)}{2}.$$
Similarly, consider the subsheaf G consisting of sections of E on C_2' vanishing at P extended by zero on C_1'. Then,
$$\frac{\chi(E_{|C_2'})}{2 - 2w_1'} = \frac{\chi(G)}{2w_2'} \le \frac{\chi(E)}{2}.$$
As in this case, $\chi(E) = \chi(E_{|C_1'}) + \chi(E_{|C_2'})$, the two inequalities put together give
$$w_1' \chi(E) \le \chi(E_{C_1'}) \le w_1' \chi(E)$$
and this is impossible as, by assumption $w_1' \chi(E)$ is not an integer. This proves b). □

3. Limits on families with singular fiber

We now want to see how to apply the results above to our situation. Assume that we have family of curves $\pi : \mathcal{C} \to S$ with S one-dimensional such that the fiber over t_0 is a curve of the form 2.1 while the generic fiber is non-singular. Note that the structure of the special fiber is preserved by finite base-change and resolution of singularities.

Assume that we have a family of rank two vector bundles $\mathcal{E}$ on $\mathcal{C} - \mathcal{C}_0 \rightarrow S - \{t_0\}$ such that the determinant of the restriction to the fibers is the canonical line bundle on the fibers. Choose a line-bundle $\mathcal{L}$ defined on $\mathcal{C} \rightarrow S$ such that the Euler-Poincaré characteristic χ of $\mathcal{E} \otimes \mathcal{L}$ restricted to each fiber is non-zero. Choose a polarization $\{w_i\}$ associated to the components C_i of $\mathcal{C}_{t_0}$ so that $\sum_{i \in I} w_i \chi \notin \mathbf{Z}$ for all $I \subset \{1, \ldots M\}$. Using the arguments of Seshadri (cf [S]) , replacing a single curve by a family with enough sections, one has a moduli space of $\{w_i\}$-semistable torsion-free sheaves on $\mathcal{C} \rightarrow S$. By properness, $\mathcal{E}$ gives rise to a map from S to this moduli space. There exists an étale covering of the moduli space over which a universal family exists. Therefore, up to replacing $\mathcal{C} \rightarrow S$ with a suitable covering, we can assume that $\mathcal{E}$ can be extended to a semistable torsion-free sheaf on the family of curves that we shall still denote by $\mathcal{E}$. If the sheaf is not locally free on $\mathcal{C}_{t_0}$, one can blow up some of the nodes of $\mathcal{C}_{t_0}$ to obtain a new surface

$$
\begin{array}{ccc}
\mathcal{C}' & \rightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
S' & \rightarrow & S
\end{array}
$$

on which the sheaf $\mathcal{E}$ can be extended to a sheaf $\mathcal{E}'$ that is locally free. In this process, one adds a few rational curves among the nodes of $\mathcal{C}_{t_0}$ and therefore, the central fiber is still of the type given in 2.1. Moreover, from 2.3 b), the restriction of the vector bundle to these new components is either trivial or of the form $\mathcal{O} \oplus \mathcal{O}(1)$ (see [G] , section 5 or [X] 1.7). Denote still with $\mathcal{L}$ the pull-back of the original $\mathcal{L}$ to the new families obtained by base-change and blow-up. Now $\mathcal{E}' \otimes \pi^*(\mathcal{L}^{-1})$ restricted to the central fiber can be taken as the limit of the original $\mathcal{E}$ and satisfies the conditions in 2.3.

We assume in what follows that we are in the framework we just described, namely we have a family of curves $\mathcal{C} \rightarrow S$ where S is the spectrum of a discrete valuation ring. Denote by η the generic point in S, by t a generator of the maximal ideal of the ring and by ν the discrete valuation.

We assume that $\mathcal{C}_\eta$ is non-singular while the central fiber is a chain of curves as described in 2.1.

We want to consider linear series for vector bundles of rank two. On the central fiber, one obtains then a limit linear series in the sense of [T1]. For the convenience of the reader, we shall reproduce the definition here.

3.1. Definition. Limit linear series *A limit linear series of rank r, degree d and dimension k on a chain of M (not necessarily rational*

and elliptic) curves consists of data I,II below for which data III, IV exist satisfying conditions a-c.

I) For every component C_i, a vector bundle E_i of rank r and degree d and a k-dimensional space of sections V_i of E_i.

II) For every node obtained by gluing Q_i and P_{i+1} an isomorphism of the projectivisation of the fibers $(E_i)_{Q_i}$ and $(E_{i+1})_{P_{i+1}}$

III) A positive integer a

IV) For every node obtained by gluing Q_i and P_{i+1}, bases $s_{Q_i}^t, s_{P_{i+1}}^t, t = 1...k$ of the vector spaces V_i and V_{i+1}

Subject to the conditions

a) $\sum_{i=1}^{M} d_i - r(M-1)a = d$

b) The orders of vanishing at P_{i+1}, Q_i of the sections of the chosen basis satisfy $ord_{P_{i+1}} s_{i+1}^t + ord_{Q_i} s_i^t \geq a$

c) Sections of the vector bundles $E_i(-aP_i), E_i(-aQ_i)$ are completely determined by their value at the nodes.

For the reader who is unfamiliar with the concept of limit linear series, this is to be understood in the following way: On a family of curves, fix a vector bundle on the generic fiber and extend it (after some base change and normalizations) to a vector bundle on the central fiber. This limit vector bundle on the central fiber is not unique. We could modify it by for example tensoring the bundle on the whole curve $\mathcal{C}$ with a line bundle with support on the central fiber. This would leave the vector bundle on the generic curve unchanged but would modify the vector bundle on the reducible curve (by adding to the restriction to each component a linear combination of the nodes). In this way, for each component C_i, one can choose a version $\mathcal{E}_i$ of the limit vector bundle such that the sections of $\mathcal{E}_i$ that vanish at some of the nodes of a component $C_j, i \neq j$ are in fact identically zero on C_j. If a vector bundle $\mathcal{E}$ was $\{w_i\}$-semistable, these modified versions no longer are. But the restriction to each component preserves the same structure . Therefore, results as in 2.3 are still valid.

As we are allowed to tensor with line bundles supported on the components of the central fiber, we can assume that $deg(\mathcal{E}_{i|C_i}) = 2g - 2 + \epsilon_i$, $-1 \leq \epsilon_i \leq 1, \sum \epsilon_i = 0$. These $\mathcal{E}_i$ are not independent of each other. Each $\mathcal{E}_{i+1}$ can be obtained from $\mathcal{E}_i$ as $\mathcal{E}_{i+1} = \mathcal{E}_i(-aF_i)$ where F_i is the union of the components $C_{i+1}, ...C_M$. In our case, a can be taken to be $g - 1$

Of special interest to us is the limit of the canonical linear series . This has rank one and is considered in [EH] [W]. The restriction of the

corresponding line bundle to a component C_i is given by

$$\omega_{i|C_i} = \mathcal{O}(2(\sum_{k \leq i} g(C_k)) - 1)P_i + 2(g - (\sum_{k \leq i} g(C_k)))Q_i).$$

In the case of a rational curve, this can be written more simply as $\mathcal{O}(2g - 2)$.

Assume now that we have a limit linear series with canonical determinant. Write $\mathcal{E}_{i|C_i} = E_i$ and write $deg E_i = 2g - 2 + \epsilon_i, \quad -1 \leq \epsilon_i \leq 1$ as above.

Then,

$$det E_i((\sum_{j < i} \epsilon_j)P_i - (\sum_{j \leq i} \epsilon_j)Q_i) =$$

$$= \mathcal{O}(2(\sum_{j \leq i} g(C_j)) - 1)P_i + 2(g - (\sum_{j \leq i} g(C_j)))Q_i)$$

Hence,

$$det E_i = \mathcal{O}((2(\sum_{j \leq i} g(C_j)) - 1) - \sum_{j < i} \epsilon_j)P_i + (2(g - (\sum_{j \leq i} g(C_j))) + \sum_{j \leq i} \epsilon_j))Q_i).$$

As each E_i is either semistable or a direct sum of two line bundles whose degrees differ in one unit, the sections of $E_i(-(g-1)P_i,)$, $E_i(-(g-1)Q_i)$ are completely determined by their values at the nodes. We then take $a = g - 1$ in the definition of limit linear series and the condition

$$\sum_i deg E_i - 2a(M - 1) = 2g - 2$$

is satisfied.

The orders of vanishing at a point P_i (or Q_i) of a linear series will be denoted by $(a_j(P_i))$,(resp $(a_j(Q_i))$, $j = 1...k$. Note that each a_j will appear at most twice and when this happens, there is a two dimensional space of sections with this vanishing at P_i(resp Q_i).

4. VANISHING AT THE NODES OF ELEMENTS IN THE KERNEL

The following results are analogous to 1.2,1.3 of [EH]. The proof of the first is almost identical to the one in [EH] and is omitted.

Here t denotes the parameter in the discrete valuation ring and ν its valuation. Choose a component C_i and denote by $\mathcal{E}_i$ as in the previous section, the vector bundle on $\pi : \mathcal{C} \to S$ whose restriction to all components of the central fiber except C_i has trivial sections.

4.1. Lemma. *For every component C_i, there is a basis $\sigma_j, j = 1...k$ of $\pi_* \mathcal{E}_i$ such that*
a) $ord_{P_i}(\sigma_j) = a_j(P_i)$
b) for suitable integers α_j, $t^{\alpha_j} \sigma_j$ are a basis of $\pi_*(\mathcal{E}_{i+1})$

4.2. Proposition *Let σ_j be a basis of $\pi_* \mathcal{E}_{C_i}$ such that $t^{\alpha_j}\sigma_j$ is a basis of $\pi_*(\mathcal{E}_{i+1})$. Then , the orders of vanishing of the σ_j at the nodes satisfy*

a) $ord_{P_i}(\sigma_j) \leq g - 2 - ord_{Q_i}\sigma_j \leq \alpha_j - 1 \leq ord_{P_{i+1}}t^{\alpha_j}\sigma_j - 1$ if E_i is an indecomposable vector bundle of degree $2g - 3$.

b) $ord_{P_i}(\sigma_j) \leq g - 1 - ord_{Q_i}\sigma_j \leq \alpha_j \leq ord_{P_{i+1}}t^{\alpha_j}\sigma_j$ if E_i is an indecomposable vector bundle of degree $2g - 1$, a direct sum of two line bundles of degree $g-1$, an indecomposable vector bundle of degree $2g-2$ or a direct sum of two line bundles of degrees $g - 1, g - 2$.

c) $ord_{P_i}(\sigma_j) \leq g - ord_{Q_i}\sigma_j \leq \alpha_j + 1 \leq ord_{P_{i+1}}t^{\alpha_j}\sigma_j + 1$ if E_i is a direct sum of two line bundles of degrees $g - 1, \ g$.

Moreover, if equality holds, then σ_j vanishes only at P_i, Q_i as a section of E_i.

Proof. The proof is similar to the proof of the analogous result in Prop 1.1 in [EH].

For a line bundle of degree d , the sum of the vanishing at two given points is at most d and if equality holds, the section does not vanish at any other point.

Similarly (see [A]), for an indecomposable vector bundle of degree $2d + 1$ or $2d$ on an elliptic curve, a section vanishes at two given points with orders adding to at most d and again, if equality holds, the section does not vanish anywhere else. This gives the first inequality in each of a,b,c above.

For the second inequality, note that $t^{\alpha_j}\sigma_j$ is a section of $\pi_*(\mathcal{E}_{i+1}) = \pi_*(\mathcal{E}_i(-(g - 1)F_{i+1}))$ where F_{i+1} represents the divisor $C_{i+1} \cup .. \cup C_M$. Therefore, $\sigma_{j|C_i}$ vanishes as a section of $\pi_*(\mathcal{E}_i)$ to order at least $g-1-\alpha_j$ along F_{i+1}. Hence, $\sigma_{j|C_i}$ vanishes to order at least $g - 1 - \alpha_j$ at Q_i. This gives the second of the inequalities.

For the last inequality, use the fact that $\mathcal{E}_{i+1} = \mathcal{E}_i$ locally along $C - F_{i+1}$. Hence, $t^{\alpha_j}\sigma_j$ vanishes as a section of $\pi_*(\mathcal{E}_{i+1})$ to order at least α_j on $C - F_{i+1}$. Hence, $t^{\alpha_j}\sigma_j$ vanishes to order at least α_j at P_{i+1}. $\square$

4.3. Remark

The inequalities above can be equalities only if there is a section whose order of vanishing at P_i, Q_i adds up to the maximum possible. We describe next when this happens for each of the possible structures of the restriction of the vector bundle to C_i. We are assuming that C_i is elliptic and P_i, Q_i are generic and in particular $\mathcal{O}(P_i - Q_i)$ is not a torsion point of the Jacobian.

If E is indecomposable of degree $2(g - 1) + 1$ (resp. $2(g - 1) - 1$), there is a finite number of sections (up to a constant) such that $ord_{P_i}(s) + ord_{Q_i}(s) = g - 1$ (respectively $g - 2$)) . In fact, for each

possible vanishing a_i at $P_i, 0 \leq a_i \leq g - 1$ (resp. $0 \leq a_i \leq g - 2$) there is one such section and no two of them are sections of the same line bundle. This follows from the fact that $h^0(E_i(-aP - (d - a)Q)) = 1$ if $deg(E_i) = 2d + 1$.

In the case of an indecomposable vector bundle of degree $2(g - 1)$, there is at most one section with sum of vanishings at the nodes being $g - 1$.

If $E = L_g \oplus L_{g-1}$, there is at most one section that vanishes with maximum sum of vanishings g. There is at most one section of L_{g-1} such that the sum of vanishing at the nodes is $g - 1$. For every value a, there is one section of L_g with vanishing a at P and $g - 1 - a$ at Q.

If E is the direct sum of two line bundles of degree $g - 1$, there are at most two independent sections with maximum vanishing at the nodes adding up to $g - 1$.

We now want to define the order of vanishing of a section ρ of $S^2(\pi_*(\mathcal{E}_i))$ as follows: Let σ_j be a basis of $\pi_*\mathcal{E}_i$ such that their orders of vanishing are the orders of vanishing of the linear series at P_i. Write $\rho = \sum_{j \leq l} f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$ with f_{jl} functions on the discrete valuation ring.

4.4. Definition *We say $ord_{P_i}\rho \geq \lambda$ if for every j, k with $f_{j,k}(P_i) \neq 0$, $ord_{P_i}\sigma_j + ord_{P_i}\sigma_k \geq \lambda$*

As in [EH] p.278, one can identify $\pi_*\mathcal{E}_i$ and $S^2(\pi_*\mathcal{E}_i)$ with submodules of $\pi_*\mathcal{E}_\eta$ and $S^2(\pi_*\mathcal{E}_\eta)$ and also with submodules of $\pi_*\mathcal{E}_{i+1}$ and $S^2(\pi_*\mathcal{E}_{i+1})$. Let

$$\rho \in S^2(\pi_*\mathcal{E}_\eta).$$

One can then find a unique value β_i such that

$$\rho_i = t^{\beta_i}\rho \in S^2(\pi_*\mathcal{E}_i) - tS^2(\pi_*\mathcal{E}_i).$$

Then for $\alpha^i = \beta_{i+1} - \beta_i$, one has

$$\rho_{i+1} = t^{\alpha^i}\rho_i \in S^2(\pi_*\mathcal{E}_{i+1}) - tS^2(\pi_*\mathcal{E}_{i+1}).$$

The following proposition follows immediately from the definitions:

4.5. Proposition. *Fix a component C_i. Assume that $\rho = \sum f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$ where the σ_j are a basis of $\pi_*\mathcal{E}_i$ such that $t^{\alpha_j}\sigma_j$ is a basis of $\pi_*\mathcal{E}_{i+1}$*

$$t^{\alpha^i}\rho \in S^2(\pi_*(\mathcal{E}_i)) - tS^2(\pi_*(\mathcal{E}_i)).$$

Then

$$ord_{P_i}(\rho) = min_{\{j,l|\nu(f_{jl})=0\}}ord_{P_i}(\sigma_j) + ord_{P_i}(\sigma_l)$$
$$\alpha^i = max_{\{j,l\}}(\alpha_j + \alpha_l - \nu(f_{jl})).$$

We now assume that the kernel of the Petri map is non-trivial on the generic curve. We can then find a section ρ_η in the kernel of the Petri map over the generic point. As above, we can find a ρ_i for each $i, 1 \le i \le M$, in the kernel of the map

$$S^2(\pi_*(\mathcal{E}_i)) \to (\pi_*(S^2\mathcal{E}_i))$$

with $\rho_i \notin tS^2(\pi_*\mathcal{E}_i)$.

4.6. Proposition *Let C_i be an elliptic curve such that the restriction of the vector bundle to C_i is the direct sum of two line bundles of degree $g - 1$. Then,*

$$ord_{P_{i+1}}(\rho_{i+1}) \ge ord_{P_i}(\rho_i) + 1$$

Equality above implies that the terms of ρ_i that give the vanishing at P_i can be written as

$$(\sigma' \otimes \bar\sigma' + \bar\sigma' \otimes \sigma') + (\sigma'' \otimes \bar\sigma'' + \bar\sigma'' \otimes \sigma'')$$

with σ', σ'' being two independent sections that vanish at P_i, Q_i with orders adding up to $g - 1$ (and in particular equality implies that these sections exist). Moreover, if

$$(b) \quad ord_{P_i}(\rho) \ge 2(\sum_{j<i} g(C_j)) - \sum_{j<i}\epsilon_j - 1 = \lambda_i - 1$$

then

$$ord_{P_{i+1}}(\rho_{i+1}) \ge min(ord_{P_i}(\rho_i) + 2, \lambda_i + 2).$$

When the inequalities for the order of vanishing at P_{i+1} are equalities, the terms of ρ_{i+1} that give the minimum vanishing at P_{i+1} glue with the terms of ρ_i that give the minimum vanishing at P_i.

Proof. Assume C_i is an elliptic curve and the vector bundle on C_i is of the form $L' \oplus L''$, where L', L'' are line bundles of degree $g - 1$. Then,

$$L' \otimes L'' = \mathcal{O}((2((\sum_{k\le i} g(C_k)) - 1) - \sum_{k<i}\epsilon_k)P_i + (2(g - (\sum_{k\le i} g(C_k)) + \sum_{k\le i}\epsilon_k)Q_i).$$

For ease of notation, we write this line bundle as

$$\mathcal{O}(\lambda_i P_i + \mu_i Q_i), \lambda_i + \mu_i = 2(g - 1).$$

Let $\rho = \sum f_{j,l}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$ be an element in the kernel of the Petri map written in terms of a basis of $\pi_*\mathcal{E}_i$ such that the $t^{\alpha_j}\sigma_j$ is a basis of $\pi_*\mathcal{E}_{i+1}$. Note that a single element of the form $\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j$ is not in the kernel of the Petri map. Hence, any ρ in the kernel has in its expression at least two summands of this type with the σ's appearing there being independent (we could have something like $c_{11}\sigma_1 \otimes \sigma_1 + c_{22}\sigma_2 \otimes \sigma_2$ but not something like $c_{12}(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1) +$

$c_{13}(\sigma_1 \otimes \sigma_3 + \sigma_3 \otimes \sigma_1)$ as the latter could be written as $\sigma_1 \otimes (c_{12}\sigma_2 + c_{13}\sigma_3) + (c_{12}\sigma_2 + c_{13}\sigma_3) \otimes \sigma_1$.

Note that $ord_{P_i}(\sigma_j) \leq \alpha_j \leq ord_{P_{i+1}} t^{\alpha_j}\sigma_j$ with the first inequality being an equality for at most two independent sections σ', σ'' that vanish at P_i, Q_i with orders adding up to $g-1$ (see 4.2 4.3).

Hence, $ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 1$ and $ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 2$ except in the case when the expression of ρ that gives the vanishing at P_i contains exactly two summands and one of the sections appearing in each summand is one of the two sections σ', σ'' named above. So, we can write the restriction of ρ to C_i as

$$\rho = (\sigma' \otimes \bar{\sigma}' + \bar{\sigma}' \otimes \sigma') + (\sigma'' \otimes \bar{\sigma}'' + \bar{\sigma}'' \otimes \sigma'') + ...$$

where the dots stand for sections with higher vanishing at P_i. Using the decomposition of E into a direct sum and up to replacing $\bar{\sigma}'$ and $\bar{\sigma}''$ by linear combinations of themselves, assume that $\sigma' = (\sigma'_1, 0)$ and $\sigma'' = (0, \sigma''_2)$. Write then $\bar{\sigma}' = (\bar{\sigma}'_1, \bar{\sigma}'_2)$ (and similarly for $\bar{\sigma}''$) in terms of the decomposition of E_i. Using that ρ is in the kernel of the Petri map, we obtain $\bar{\sigma}'_1 = 0$, $\bar{\sigma}''_2 = 0$. Hence, the order of vanishing of $\bar{\sigma}'$ is in fact the order of vanishing of $\bar{\sigma}'_2$. From the fact that $\sigma'_1 \sigma''_2$ is a section of $L'L''$ vanishing to order adding up to $2g-2$ between the two nodes, $ord_{P_i}(\sigma'\sigma'') = \lambda_i$ and $ord_{P_i}(\sigma'\bar{\sigma}') \neq ord_{P_i}(\sigma'\sigma'')$. Assume now that $ord_{P_i}(\rho_i) \geq \lambda_i$. Then

$$\lambda_i \leq ord_{P_i}(\rho_i) = ord_{P_i}(\sigma') + ord_{P_i}(\bar{\sigma}') \neq$$

$$\neq ord_{P_i}(\sigma') + ord_{P_i}(\sigma'') = \lambda_i.$$

Hence

$$\lambda_i + 1 \leq ord_{P_i}(\rho_i) = ord_{P_i}(\sigma') + ord_{P_i}(\bar{\sigma}').$$

On the other hand by 4.2 4.5,

$$ord_{P_{i+1}}(t^{\alpha^i}\rho_i) \geq ord_{P_{i+1}}(t^{\alpha'}\sigma') + ord_{P_{i+1}}(t^{\bar{\alpha}'}\bar{\sigma}') \geq ord_{P_i}(\sigma') + ord_{P_i}(\bar{\sigma}') + 1.$$

The two inequalities together conclude the proof of the result when $ord_{P_i}(\rho_i) \geq \lambda_i$.

Assume now that $ord_{P_i}(\rho_i) = \lambda_i - 1$ and we want to show that also in this case $ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 2$.

We can again assume that the terms in ρ that gives the minimum vanishing at P_i are of the form

$$(\sigma' \otimes \bar{\sigma}' + \bar{\sigma}' \otimes \sigma') + (\sigma'' \otimes \bar{\sigma}'' + \bar{\sigma}'' \otimes \sigma'')$$

and that $\alpha^i = \alpha' + \bar{\alpha}' = \bar{\alpha}'' + \alpha'' = ord_{P_i}(\rho_i) + 1$ otherwise using 4.2 and 4.5, we would be done. Note now that $h^0(C_i, L' \otimes L''(-(\lambda_i - 1)P_i - \mu_i Q_i)) = 1$ because the degree of this line bundle is one. Also

$h^0(C_i, L' \otimes L''(-\lambda_i P_i - \mu_i Q_i)) = h^0(C, \mathcal{O}) = 1$ while $h^0(L' \otimes L''(-(\lambda_i - 1)P_i - (\mu_i - 1)Q_i)) = 2$. As

$$H^0(C, L' \otimes L''(-\lambda_i P_i - \mu_i Q_i)) \subset H^0(C, L' \otimes L''(-(\lambda_i - 1)P_i - \mu_i Q_i))$$

these two vector spaces having the same dimension, coincide and a section of $L' \otimes L''$ that vanishes at P_i with multiplicity precisely $\lambda_i - 1$ vanishes at Q_i with multiplicity at most $\mu_i - 1 = 2g - 2 - \lambda_i - 1 = 2g - 2 - (\lambda_i + 1)$. Then a section gluing with it vanishes at P_{i+1} with order at least $\lambda_i + 1 = (\lambda_i - 1) + 2$. In particular, $(t^{\alpha'}\sigma')(t^{\bar{\alpha}'}\bar{\sigma}')$ and $(t^{\alpha''}\sigma'')(t^{\bar{\alpha}''}\bar{\sigma}'')$ satisfy the condition. Then,

$$\rho_{i+1} = t^{\alpha^i}\rho_i = (t^{\alpha'}\sigma' \otimes t^{\bar{\alpha}'}\bar{\sigma}' + t^{\bar{\alpha}'}\bar{\sigma}' \otimes t^{\alpha'}\sigma') + (t^{\alpha''}\sigma'' \otimes t^{\bar{\alpha}''}\bar{\sigma}'' + t^{\bar{\alpha}''}\sigma'' \otimes t^{\alpha''}\bar{\sigma}'')$$

vanishes at P_{i+1} with the required multiplicity.

If there is equality, $\alpha^i = \alpha' + \bar{\alpha}' = \alpha'' + \bar{\alpha}''$ and the terms in ρ_{i+1} that give the vanishing at P_{i+1} glue with the terms in ρ_i that give the vanishing at P_i. □

4.7. Proposition *Assume that E_i is an indecomposable vector bundle of degree $2(g - 1)$. Then*

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 2.$$

If the inequality is an equality, the terms in ρ_{i+1} that give minimum vanishing at P_{i+1} glue with the terms in ρ_i that give minimum vanishing at P_i.

Proof. In this case there is at most one section σ that vanishes at P_i and Q_i with multiplicity adding up to $g - 1$ (see 4.3). As an element of the form $\sigma \otimes \bar{\sigma} + \bar{\sigma} \otimes \sigma$ is not in the kernel of the Petri map, in the terms of ρ_i that give minimum vanishing at P_i, $\rho_i = \sum_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j) + ...$ for at least one pair $\{\sigma_j, \sigma_l\}$ does not contain σ. Hence the inequality follows. If some of the terms on ρ_{i+1} that give the vanishing at P_{i+1} come from terms on ρ_i with vanishing at P_i at least $ord_{P_i}\rho_i + 1$, then by the same argument, $ord_{P_{i+1}}(\rho_{i+1}) \geq (ord_{P_i}(\rho_i) + 1) + 2$. This proves the second statement.

□

4.8. Proposition *Assume that E_i is an indecomposable vector bundle of degree $2(g - 1) + \epsilon$, $\epsilon = 1, -1$. Let ρ_i be as before Then either*

$$(a) \quad ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) - \epsilon + 2$$

or b)

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) - \epsilon + 1$$

and the terms in ρ_i that give the vanishing at P_i can be written in the form

$$\rho = c_{12}(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1) + c_{34}(\sigma_3 \otimes \sigma_4 + \sigma_4 \otimes \sigma_3) + ...$$

where all the σ_j are sections such that the sums of vanishing at P_i, Q_i is $g - 1$ if $\epsilon = 1$, $g - 2$ if $\epsilon = -1$ and ρ does not admit an expression with fewer than three summands $\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j$.

Also, when equality occurs, the terms in ρ_{i+1} that give minimum vanishing at P_{i+1} glue with the terms in ρ_i with minimum vanishing at P_i.

Proof. We prove it in the case $\epsilon = 1$, the other being analogous.

If $\rho_i = \sum f_{j,l}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$ is an element in the kernel, then there are at least two pairs j, l that give the minimum vanishing at each point in the above expression. If in one of these pairs at most one of the sections has vanishings at P_i, Q_i adding up to $g - 1$, using 4.2, 4.5 above, we are done. Assume then that in the above expression all the sections that appear have sum of vanishings at P_i, Q_i equal to $g - 1$. We want to check that then, the expression for ρ has at least three summands.

Choose a line bundle L' of degree $g - 1$ not linearly equivalent to $aP + (g - 1 - a)Q, 0 \le a \le g - 1$. Let L'' be the line bundle of degree g such that $L' \otimes L'' = detE$. Then, E can be deformed to a direct sum $L' \oplus L''$. On $L' \oplus L''$ there are exactly g sections s_j with sum of vanishing at P_i, Q_i being $g - 1$. In fact $ord_{P_i}(s_j) = j, ord_{Q_i}(s_j) = g - 1 - j$. As $\deg(L''(-jP_i - (g - 1 - j)Q_i)) = 1$, each one of these sections vanishes at an additional point R_j. The R_j are all different: if $R_j = R_k$, $j > k$, then from the fact that s_j, s_k are sections of the same line bundle L'', we obtain $(j - k)P_i \equiv (j - k)Q_i$ contradicting the genericity of the pair P_i, Q_i. Assume then

$$c_{i_1 i_2}(\sigma_{i_1} \otimes \sigma_{i_2} + \sigma_{i_2} \otimes \sigma_{i_1}) + c_{i_3 i_4}(\sigma_{i_3} \otimes \sigma_{i_4} + \sigma_{i_4} \otimes \sigma_{i_3})$$

is in the kernel of the Petri map. Then, $2c_{i_1 i_2}s_{i_1}s_{i_2} = -2c_{i_3 i_4}s_{i_3}s_{i_4}$. Hence these two sections vanish at the same points and we obtain $\{R_{i_1}, R_{i_2}\} = \{R_{i_3}, R_{i_4}\}$ which is a contradiction. $\square$

4.9. Proposition *Assume that C_i is elliptic and E_i is a direct sum of two line bundles of degree $g - 1$ and $g - 1 + \epsilon, \epsilon = 1, -1$ respectively . Then either*

$$ord_{P_{i+1}}(\rho_{i+1}) \ge ord_{P_i}(\rho_i) - \epsilon + 2$$

or

$$ord_{P_{i+1}}(\rho_{i+1}) \ge ord_{P_i}(\rho_i) - \epsilon + 1$$

and ρ_i can be written in the form

$$\rho_i = \sum f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j) + f'(\sigma' \otimes \sigma + \sigma \otimes \sigma') + f''(\sigma'' \otimes \sigma_i + \sigma_i \otimes \sigma")$$

where the $\sigma_j, \sigma_l, \sigma_i,$ σ' are all sections of the line bundle of higher degree.

Moreover, if

$$ord_{P_i}(\rho) \geq 2((\sum_{j \leq i} g(C_j)) - 1) - \sum_{j < i} \epsilon_j - 1,$$

then

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 2$$

or ρ can be written as before with $f' = f" = 0$. When the inequalities for the order of vanishing at P_{i+1} are equalities, the terms in ρ_{i+1} that give the minimum vanishing at P_{i+1} glue with the terms in ρ_i that give the minimum vanishing at P_i.

Proof. We assume $\epsilon = 1$, the case $\epsilon = -1$ being similar.

Write $E_i = L' \oplus L"$ where L' is a line bundle of degree g and $L"$ is a line bundle of degree $g - 1$. There are then at most one section of L' and one section of $L"$ that vanish only at P_i, Q_i with sum of vanishing at these two points being $g, g - 1$ respectively. For these two sections, one has

$$ord_{P_i}(\sigma') \leq g - ord_{Q_i}(\sigma') \leq \alpha' + 1 \leq ord_{P_{i+1}}(t^{\alpha'}\sigma') + 1$$

$$ord_{P_i}(\sigma'') \leq g - 1 - ord_{Q_i}(\sigma'') \leq \alpha'' \leq ord_{P_{i+1}} t^{\alpha''}\sigma''.$$

For any other section of L' one has

$$ord_{P_i}(\sigma'_j) \leq g - 1 - ord_{Q_i}(\sigma'_j) \leq \alpha'_j \leq ord_{P_{i+1}} t^{\alpha'_j}\sigma'_j$$

while for other sections σ''_j one has

$$ord_{P_i}(\sigma''_j) \leq g - 2 - ord_{Q_i}(\sigma''_j) \leq \alpha''_j - 1 \leq ord_{P_{i+1}} t^{\alpha''_j}\sigma''_j - 1.$$

Write now

$$\rho_i = \sum f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j).$$

As ρ_i is in the kernel of the Petri map, not all terms in this expression can contain σ'. Hence, $ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i)$.

If this expression has some σ_j or σ_l that are not paired with linear combinations of σ', σ'' or the sections σ'_j , then the vanishing of ρ_{i+1} at P_{i+1} increases by at least one unit with respect to that at P_i. If this increase does not take place, using the decomposition of E_i into direct sum, we can write

$$\rho_i = f'((\sigma', 0) \otimes (\sigma'_1, \sigma"_1) + (\sigma'_1, \sigma"_1) \otimes (\sigma', 0)) +$$
$$f"((0, \sigma'') \otimes (\sigma'_2, \sigma''_2) + (\sigma'_2, \sigma''_2) \otimes (0, \sigma'')) +$$

$$+ \sum f_{jl}((\sigma_j, 0) \otimes (\sigma_l, 0) + (\sigma_l, 0) \otimes (\sigma_j, 0).$$

From the condition of this section being in the kernel, one obtains

$$\sigma_2'' = 0, \ f'\sigma'\sigma_1'' + f''\sigma''\sigma_2' = 0.$$

If f', $f'' = 0$, we are in the case when ρ can be written only in terms of the sections of the line bundle of higher degree. If f', f'' are not both zero, then both must be non-zero by the equation above and we proceed as in the case of the sum of two line bundles of the same degree to get the result.

$\square$

4.10. Proposition *If C_i is a rational curve and E_i is the direct sum of two line bundles of degree $g - 1$, then*

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i).$$

If C_i is a rational curve and E_i is a direct sum of two line bundles of degree $g - 1, g - 1 + \epsilon$, then, either

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) - \epsilon$$

or

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) - \epsilon - 1$$

and ρ_i can be written in the form

$$\rho_i = \sum f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$$

where all of the σ_j, σ_l are sections of the line bundle of higher degree.

The proof of this last claim is similar to the previous ones and is left to the reader.

4.11. Proposition *Let C_i be elliptic and the restriction of E to C_i a direct sum of two line bundles of degree $g - 1$. If in the expression of the restriction of ρ to C_i all the sections of E that appear are linearly dependent at P_i, then*

$$ord_{P_{i+1}}(\rho_{i+1}) \geq ord_{P_i}(\rho_i) + 2$$

and equality above implies that in the expression of ρ_i there are at least three terms or that the expression for ρ_i can be written in terms of sections of a line subbundle of E_i.

Proof. The inequality has already been proved in 4.6.

Assume now that there is equality and the expression for ρ_i has precisely two terms. Because of the condition on the fibers at P_i at most one of the sections appearing in the expression for ρ_i has vanishing

at the nodes adding up to $g-1$. Call this section σ' and the line bundle it generates L. Write

$$\rho_i = (\sigma' \otimes \bar{\sigma}' + \bar{\sigma}' \otimes \sigma') + (\sigma'' \otimes \bar{\sigma}'' + \bar{\sigma}'' \otimes \sigma'').$$

Then σ'', $\bar{\sigma}''$ have order of vanishing at P_i, Q_i adding up to $g-2$. Therefore they are completely determined by their direction at P. This implies that σ'', $\bar{\sigma}''$ are also sections of the line bundle L and hence so is $\bar{\sigma}'$ by the condition of ρ being in the kernel of the Petri map. So the result is satisfied in this case.

Assume now that none of the sections involved vanishes to the maximum order $g-1$ between the two nodes. Write the sections appearing in the expression in terms of the decomposition of E_i as

$$((\sigma_1', \sigma_2') \otimes (\bar{\sigma}_1', \bar{\sigma}_2') + (\bar{\sigma}_1', \bar{\sigma}_2') \otimes (\sigma_1', \sigma_2'))+$$

$$((\sigma_1'', \sigma_2'') \otimes (\bar{\sigma}_1'', \bar{\sigma}"_2) + (\bar{\sigma}_1'', \bar{\sigma}_2'') \otimes (\sigma_1'', \sigma"_2))).$$

Each of the σ that appears in the expression above is either zero or is the unique section of L (resp L') whose order of vanishing at P_i is a given a and at Q_i is at least $g-2-a$. Then, the condition of ρ being in the kernel implies that

$$(*)\sigma_1'\bar{\sigma}_1' + \sigma_1''\bar{\sigma}_1'' = 0, \ \sigma_2'\bar{\sigma}_2' + \sigma_2''\bar{\sigma}_2'' = 0,$$

$$\sigma_1'\bar{\sigma}_2' + \sigma_2'\bar{\sigma}_1' + \sigma_1''\bar{\sigma}_2'' + \sigma_2''\bar{\sigma}_1'' = 0.$$

An argument as in the proof of 4.8 implies that

$$\sigma_1' = \lambda\sigma_1'', \ \bar{\sigma}_1'' = -\lambda\bar{\sigma}_1' \ \text{or} \ \sigma_1' = \lambda\bar{\sigma}_1'', \ \sigma_1'' = -\lambda\bar{\sigma}_1'.$$

Similarly

$$\sigma_2' = \mu\sigma_2'', \ \bar{\sigma}_2'' = -\mu\bar{\sigma}_2' \ \text{or} \ \sigma_2' = \mu\bar{\sigma}_2'', \ \sigma_2'' = -\mu\bar{\sigma}_2'.$$

Let us assume that we are in the first case for both equations (the remaining cases are treated similarly). Substituting in the third equation above, we obtain

$$(\lambda - \mu)(\sigma_1''\bar{\sigma}_2' - \sigma_2''\bar{\sigma}_1') = 0.$$

If $\lambda = \mu$, the original section was trivial against the assumption. If $\lambda \neq \mu$, then $(\sigma_1''\bar{\sigma}_2' - \sigma_2''\bar{\sigma}_1') = 0$ and an argument as in the proof of 4.8 implies that

$$\sigma_1'' = \nu\sigma_2'', \ \bar{\sigma}_1' = \nu\bar{\sigma}_2' \ \text{or} \ \sigma_1'' = \nu\bar{\sigma}_1', \ \sigma_2'' = \nu\bar{\sigma}_2'.$$

In both cases, the original section was trivial against the assumptions. $\qquad\square$

4.12. Proposition *Let C be an elliptic curve and assume that E_i is an indecomposable vector bundle of degree $2g - 1$. Consider the sections of the vector bundle that vanish at P with multiplicity a and at Q with multiplicity $g - 1 - a$. Then, the fibers of these sections at P are different except maybe for some pairs a_1, a_2 and if this happens then $\det E_i = \mathcal{O}((a_1 + a_2)P + (2g - 2 - a_1 - a_2)Q)$.*

Proof. Consider two line subbundles of E_i generated by these sections, say

$$\mathcal{O}(a_1 P + (g - 1 - a_1)Q) \to E_i$$
$$\mathcal{O}(a_2 P + (g - 1 - a_2)Q) \to E_i.$$

Assume $a_1 > a_2$ and write $a' = a_1 - a_2$, $b' = g - 1 - a_2$, $E' = E_i(-a_2 P)$ tensoring the spaces in the two maps above with $\mathcal{O}(-a_2 P)$, we obtain

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(a'P + (b' - a')Q) & \to & E' & \to & L & \to & 0 \\
& & & & \uparrow & & & & \\
& & & & \mathcal{O}(b'Q) & & & &
\end{array}
$$

As the vertical arrow does not factor through $\mathcal{O}(a'P + (b' - a')Q)$, it gives rise to a non-zero morphism to L (which is a line bundle of degree $b' + 1$). If this map is zero at P_i, then $L = \mathcal{O}(P_i + b'Q_i)$. In particular E_i is as claimed and a_2 is determined by E_i and a_1. $\qquad\square$

5. Proof of the Theorem

5.1. Proposition *For a section ρ in the kernel of the Petri map, one has one of the three options below*

a)
$$ord_{P_i}(\rho) \geq 2(\sum_{k<i} g(C_k)) - \sum_{k<i} \epsilon_k$$

b)
$$ord_{P_i}(\rho) \geq 2(\sum_{k<i} g(C_k)) - \sum_{k<i} \epsilon_k - 1$$

and on C_{i-1} the equation for the terms of ρ_t giving the maximum vanishing at Q_{i-1} is of the form

$$\sum_{jl} f_{jl}(\sigma_j \otimes \sigma_l + \sigma_l \otimes \sigma_j)$$

cannot be written with fewer than three summands and the σ appearing in the expression vanish only at the two nodes.

c)
$$ord_{P_i}(\rho) \geq 2(\sum_{k<i} g(C_k)) - \sum_{k<i} \epsilon_k - 1$$

and on C_{i-1} the equation for the terms of ρ_t giving the maximum vanishing at Q_{i-1} can be written in terms of sections of a line subbundle of $C_{i-k} \cup ... \cup C_{i-1}$ of degree $\frac{\deg(E_{i-k})+...+\deg(E_{i-1})}{2} + 1$

Proof. By induction on i, the case $i = 1$ being clear as

$$2(\sum_{k<1} g(C_k)) - \sum_{k<1} \epsilon_k = 0$$

Assume the result correct for $i - 1$ and show it for i. If condition a) holds, then the result can be obtained from the propositions in the previous paragraph.

If condition b) holds, and C_i is rational or elliptic with E the direct sum of two line bundles, then the result follows also from the previous propositions.

Assume that C_i is elliptic and E_i is an indecomposable vector bundle of odd degree. If the vanishing at P_i increases by at least one unit, we are back to conditions a) or b). Otherwise, the restriction of ρ to C_i has at least three terms. With a suitable choice of gluing of Q_i, P_{i+1} up to two arbitrary directions can be made to agree. By the independent genericity of C_{i-1}, C_i not all directions involved in ρ_i will glue with those in ρ_{i-1}. Hence, this is impossible.

Assume now that condition c) holds. If C_i is elliptic and E_i is an indecomposable vector bundle, we reason as in case b). If C_i is elliptic or rational and E_i is the direct sum of two line bundles of the same degree, then 4.6, 4.10 and 4.11, above show that either a) or c) holds again. If C_i is elliptic or rational and E_i is the direct sum of two line bundles of different degrees, then by stability, the line bundle of higher degree cannot glue with a line bundle of higher degree in the previous component. Hence, the result is satisfied again.

$\square$

The Theorem now follows from the statement above: on the last component (that we can assume to be rational), the vector bundle is the direct sum of two line bundles of the same degree $g - 1$ and $\sum \epsilon_j = 0$. The proposition above implies that ρ vanishes at P_g with multiplicitly at least $2g - 1$ and this is impossible as this multiplicity is at most $2g - 2$. Hence ρ cannot exist and the Petri map is injective.

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